Maths problem sheet
BeyondResearch application
Deadline: 16/09/2023 23:59 GMT
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Instructions:

- Please write clearly and be detailed in your working and explanations to help us get an impression how you tackle problems and how you think. *This assessment is more about the way you get to the answer, than the actual answer itself!*
- You should approach every problem with the curiosity and sharp mind of a researcher trying to uncover the hidden secrets behind the veil.
- Start each problem on a new page. Each page of your working should have the number of the problem you are trying to answer on top of the page, so we can put your writing in the right context.
- Note that some of the problems are not your typical math questions and we would like to see your creativity when engaging with them. Indeed, some questions are open ended, and do not have just one definite answer.
- *Don't worry if you struggle with a problem and cannot solve it.* Write down your ideas, your working up to the point where you are stuck, what you think the next step is and why you are stuck. If you find more than one way of approaching the problem, write it down, too! We are interested to see how you approach a problem and how you develop your thinking. You will be assessed holistically.

Let's warm up!

Problem 1. What is the relationship between the figure given by the locus of the equation $2x-2y = 2$ and the figure given by the locus of the equation $23x^2 - 46x + 23y^2 + 92y + 69 = 0$? Answer in full detail and justify your reasoning.

Problem 2. Using differentiation rules find the derivative of $\tan(x)$. Give it in its most simplified form. (You may use that the derivative of $\sin(x)$ is $\cos(x)$, but all other derivatives have to be derived.)

Problem 3. Discuss how many solutions the following system of equations has:

$$x_1 - x_2 + 2x_3 + 2 = 0 \tag{1}$$
$$2x_2 - x_3 - 5 = 0 \tag{2}$$
$$3x_1 - x_2 + 5x_3 + 1 = 0 \tag{3}$$

Problem 4. A remaining piece of cardboard has the form of a standard parabola. The length from the vertex to the edge is $24\,cm$. We wish to cut out a rectangle from this piece with the greatest possible area. What are the width and the height of this rectangle, and what is its area?

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Let's go abstract!

Problem 5. 2D Vectors are typically depicted as arrows in the plane. Does the position of the arrow matter? Discuss.

Problem 6. Besides arrows, vectors are often represented as lists of numbers. Whereas vectors as arrows are typically used in Physics, vectors as lists are used in Computer Science. Which of those viewpoints on vectors is the right one and why? Discuss.

Problem 7. Investigate how one could multiply vectors in a meaningful way and give one or several possible definitions of products of vectors in the plane. Justify your choice and explain your reasoning.

Problem 8. We wish to do formal addition and subtraction with a set of three stones. For this purpose we shall model the stones by three letters $S1$, $S2$ and $S3$. Our aim is to develop a formal algebra of stones that allows us to do addition and subtraction as we do with integers, so expressions like $S1 + S2$, or $-5 \times S3$ should be possible.

(a) What could be a sensible intended interpretation of $S1 + S2$?

(b) What would be a sensible interpretation of $-5 \times S3$?

(c) Develop ideas how such a formal algebra could be defined and implemented.

(d) Can you solve linear and quadratic equations in your formal algebra of stones? Discuss.

(e) What relationship can you draw from your formal algebra of stones to working with numbers like the reals, say?

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Now things get slightly strange!

Welcome to a parallel mathematical universe, *as we will now alter the rules of the game ever so slightly!* You will be introduced to a new mathematical theory in the form of a new set of rules. When tackling the following problems you should be operating within this set of rules, *so be careful!*

We introduce a new set of numbers R, which have the following properties:

(I) R contains the real numbers $\mathbb{R}$; these are all integers $\ldots-2,-1,0,1,2,\ldots$, all rational numbers, i.e. fractions of integers, as well as irrational numbers like $\sqrt{2}$, π, e etc. You can treat them all as you are already used to.

(II) R also contains numbers d with the property that $d^2=0$. The set of all the numbers d in R that square to zero is denoted by D.

(III) You can add and multiply all numbers in R like you are used to with the integers; *but you will have to be careful with division!*

(IV) We have the following mysterious property: For every function $f: R\to R$ and for every x_0 in R there are unique numbers a_{x_0} and b_{x_0} in R such that

$$f(x_0+d)=a_{x_0}+b_{x_0}\cdot d$$

for all d in the set D.

Problem 9. Show the following:

(a) For any number λ in R and d in D the product λd is again in D.

(b) For any two numbers d_1 and d_2 in D we have that $(d_1+d_2)^3=0$.

(c) Show that if $d_1d_2=0$ for all numbers d_1 and d_2 in D, then the set D has only one element, namely 0. How could you use this fact to show that for arbitrary two numbers d_1 and d_2 in D their sum d_1+d_2 does not necessarily belong to D?

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We shall call the unique number b_{x_0} in rule (IV) the *mysterious number* of f at x_0 and denote it by $f'(x_0)$; so $f'(x_0) = b_{x_0}$ by definition. As the mysterious number exists for every number x_0, we get a new function $f' : R \to R$ that maps each number x in R to the *mysterious number* $f'(x)$ of f at x. We shall refer to the function f' as the *mysterious function*. ***Let's study these mysterious numbers and functions to see whether we can uncover some of their mystery and make some sense out of them!***

Problem 10. Let x_0 be an arbitrary but fixed number in R and $f : R \to R$ a function.

(a) Show that the the number a_{x_0} given in rule (IV) is equal to $f(x_0)$; hence conclude that the equation in rule (IV) can be also written like this:

$$f(x_0 + d) = f(x_0) + f'(x_0)d$$

(b) Let's turn more concrete and consider the function f with $f(x) = x^2$. Show that $f'(x_0) = 2x_0$ and hence

$$f(x_0 + d) = x_0^2 + 2x_0 d$$

for all numbers d in D. What could this mean? Can you think of a geometric interpretation of this equation?

(c) Now let's study the function f with $f(x) = x^3$. Show that $f'(x_0) = 3x_0^2$ and hence

$$f(x_0 + d) = x_0^3 + 3x_0^2 d$$

for all numbers d in D. What could this mean? Can you think of a geometric interpretation of this equation?

(d) Looking back at what you have learned in this problem, can you find an interpretation for the mysterious number $f'(x_0)$ and the mysterious function f'? Explain your thinking and your reasons.

That's All Folks!

We hope you had fun exploring, thinking about and solving the problems.
(P.S. The parallel mathematical universe we have introduced you to truly exists... (TBC) ;-))